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## Removability of level sets for two classes of fully nonlinear equations

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### 1 Introduction

In the early 20th century, Radó [17] proved the following theorem for complex analytic functions.

**Theorem 1.1.** *Let  $f$  be a continuous complex-valued function in a domain  $\Omega \subset \mathbb{C}$ . If  $f$  is analytic in  $\Omega \setminus f^{-1}(0)$ , then  $f$  is actually analytic in the whole domain  $\Omega$ .*

This result says that a level set is always removable for continuous analytic functions. Later, an analogous result of Radó's result for harmonic functions has been obtained.

**Theorem 1.2.** [1, 8] *Let  $u$  be a real-valued continuously differentiable function defined in a domain  $\Omega \subset \mathbb{R}^n$ . If  $u$  is harmonic in  $\Omega \setminus u^{-1}(0)$ , then it is harmonic in the whole domain  $\Omega$ .*

Such removability problems have been intensively studied. The corresponding results for linear elliptic equations were proved by Šabat [18]. The case of  $p$ -Laplace equation has been treated in [12, 14]. Juutinen and Lindqvist [13] proved the removability of a level set for *viscosity* solutions to general quasilinear elliptic and parabolic equations. Recently, we have obtained this type of removability results for general *fully nonlinear* degenerate elliptic and parabolic equations which cover most of the previous results [21]. In Section 2, we shall focus on the removability of a level set for solutions to fully nonlinear equations.

These results stated above concerns the removability of the inverse image of "one point." One may consider the following extension: How about the removability of  $u^{-1}(E)$  for general subset  $E \subset \mathbb{R}$  rather than one point? This type of removability result has been studied by Král [15] for Laplace equation  $\Delta u = 0$ .

**Theorem 1.3.** [15] *Let  $u$  be a real-valued continuously differentiable function defined in a domain  $\Omega \subset \mathbb{R}^n$  and  $E$  a subset of  $\mathbb{R}$ . We suppose that each compact subset  $F$  of  $E$  is at most countable. If  $u$  is harmonic in  $\Omega \setminus u^{-1}(E)$ , then it is harmonic in the whole domain  $\Omega$ .*

In this article, we shall obtain Král type removability theorems for two classes of elliptic fully nonlinear equations. The equations which we deal with are so-called  $k$ -Hessian equations and  $k$ -curvature equations.

This article is organized as follows. In the following section, we review our previous results, which say that a level set is always removable for solutions to fully nonlinear elliptic or parabolic equations under some assumptions. In section 3, we give the definition of “generalized solutions” to  $k$ -Hessian equations and  $k$ -curvature equations, and state our main theorem, Král type removability result. The proof of the main theorem is given in Section 4.

## 2 Radó type removability result for solutions to fully nonlinear PDEs

In this section, we consider the removability of a level set for solutions to fully nonlinear equations, which has been already proved in [21]. The equations which we are concerned with are the following degenerate elliptic, fully nonlinear equation

$$F(x, u, Du, D^2u) = 0, \quad (2.1)$$

in  $\Omega \subset \mathbb{R}^n$ , or the parabolic one

$$u_t + F(t, x, u, Du, D^2u) = 0, \quad (2.2)$$

in  $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$ . In both equations,  $D$  means the derivation with respect to the space variables, that is,

$$Du := \left( \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)^T, \quad D^2u := \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}. \quad (2.3)$$

Here  $A^T$  denotes the transpose of a matrix  $A$ .

We use the following notations in this article.

- $\mathbb{S}^{n \times n} := \{n \times n \text{ real symmetric matrix}\}.$
- For  $X, Y \in \mathbb{S}^{n \times n}$ ,  $X \leq Y \stackrel{\text{def}}{\iff} Y - X$  is non-negative definite.  
(i.e.,  $(Y - X)\xi \cdot \xi \geq 0$  for all  $\xi \in \mathbb{R}^n$ .)
- For  $\xi, \eta \in \mathbb{R}^n$ ,  $\xi \otimes \eta$  denotes the  $n \times n$  matrix with the entries

$$(\xi \otimes \eta)_{ij} = \xi_i \eta_j \quad (i, j \in \{1, \dots, n\}). \quad (2.4)$$

- For  $x \in \mathbb{R}^n$  and for  $r > 0$ ,

$$B_r(x) := \{z \in \mathbb{R}^n \mid |z - x| < r\}. \quad (2.5)$$

- Let  $\Omega$  be an open set in  $\mathbb{R}^n$  or  $\mathbb{R} \times \mathbb{R}^n$ .

$$\text{USC}(\Omega) := \{u : \Omega \rightarrow [-\infty, \infty), \text{ upper semicontinuous}\}, \quad (2.6)$$

$$\text{LSC}(\Omega) := \{u : \Omega \rightarrow (-\infty, \infty], \text{ lower semicontinuous}\}. \quad (2.7)$$

To deal with our problem, we consider the class of *viscosity solutions*, which are solutions in a certain weak sense. The theory of viscosity solutions to fully nonlinear equations was developed by Crandall, Evans, Ishii, Jensen, Lions and others. See, for example, [6, 7, 9, 11]. In many nonlinear partial differential equations, the viscosity framework allows us to obtain existence and uniqueness results under mild hypotheses. Here we recall the notion of viscosity solutions to the fully nonlinear elliptic equations (2.1).

**Definition 2.1.** Let  $\Omega$  be a domain in  $\mathbb{R}^n$ .

- (i) A function  $u \in \text{USC}(\Omega)$  is said to be a *viscosity subsolution* to (2.1) in  $\Omega$  if  $u \not\equiv -\infty$  and for any function  $\varphi \in C^2(\Omega)$  and any point  $x_0 \in \Omega$  which is a maximum point of  $u - \varphi$ , we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0. \quad (2.8)$$

- (ii) A function  $u \in \text{LSC}(\Omega)$  is said to be a *viscosity supersolution* to (2.1) in  $\Omega$  if  $u \not\equiv \infty$  and for any function  $\varphi \in C^2(\Omega)$  and any point  $x_0 \in \Omega$  which is a minimum point of  $u - \varphi$ , we have

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0. \quad (2.9)$$

- (iii) A function  $u \in C^0(\Omega)$  is said to be a *viscosity solution* to (2.1) in  $\Omega$  if it is both a viscosity subsolution and supersolution to (2.1) in  $\Omega$ .

We omit the proof of the following proposition. We say that  $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n \times n} \rightarrow \mathbb{R}$  is degenerate elliptic if

$$F(x, r, q, X) \geq F(x, r, q, Y) \quad (2.10)$$

for every  $x \in \Omega$ ,  $r \in \mathbb{R}$ ,  $q \in \mathbb{R}^n$ ,  $X, Y \in \mathbb{S}^{n \times n}$  with  $X \leq Y$ .

**Proposition 2.2.** Let  $\Omega$  be a domain in  $\mathbb{R}^n$  and suppose that  $F = F(x, r, q, X)$  is continuous and degenerate elliptic. If a  $C^2$  function  $u$  is a classical solution to  $F(x, u, Du, D^2u) = 0$ , then it is a viscosity solution to the same equation.

Here we state the result concerning the removability of a level set for solutions to (2.1).

**Theorem 2.3.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$ . We suppose that  $F = F(x, r, q, X)$  satisfies the following conditions.*

(A1)  *$F$  is a continuous function defined in  $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n \times n}$ .*

(A2)  *$F$  is degenerate elliptic.*

(A3)  *$F(x, 0, 0, O) = 0$  for every  $x \in \Omega$ .*

(A4) *There exists a constant  $\alpha > 2$  such that for every compact subset  $K \Subset \Omega$  we can find positive constants  $\varepsilon, C$  and a continuous, non-decreasing function  $\omega_K : [0, \infty) \rightarrow [0, \infty)$  which satisfy  $\omega_K(0) = 0$  and the following:*

$$\begin{aligned} F(y, s, j|x - y|^{\alpha-2}(x - y), Y) - F(x, r, j|x - y|^{\alpha-2}(x - y), X) \\ \leq \omega_K(|r - s| + j|x - y|^{\alpha-1} + |x - y|) \end{aligned} \quad (2.11)$$

whenever  $x, y \in K$ ,  $r, s \in (-\varepsilon, \varepsilon)$ ,  $j \geq C$ ,  $X, Y \in \mathbb{S}^{n \times n}$  and

$$\begin{aligned} -3j(\alpha - 1)|x - y|^{\alpha-2}I_{2n} &\leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \\ &\leq 3j(\alpha - 1)|x - y|^{\alpha-2} \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} \end{aligned} \quad (2.12)$$

holds.

If  $u \in C^1(\Omega)$  is a viscosity solution to (2.1) in  $\Omega \setminus u^{-1}(0)$ , then  $u$  is a viscosity solution to (2.1) in the whole domain  $\Omega$ .

**Remark 2.1.** We remark about the regularity assumption on  $u$ . This theorem also holds if we only assume that  $u$  is continuously differentiable on some neighborhood of  $\{u = 0\}$  instead of assuming that  $u \in C^1(\Omega)$ . However, one can not weaken the differentiability assumption. More precisely, if we replace  $u \in C^1(\Omega)$  by  $u \in C^{0,1}(\Omega)$ , the conclusion fails to hold. Define the function  $u$  by

$$u(x) = |x_1|, \quad x = (x_1, \dots, x_n) \in \Omega = B_1 = \{|x| < 1\}. \quad (2.13)$$

It is easily checked that  $u$  satisfies  $-\Delta u = 0$  in  $\Omega \setminus u^{-1}(0) = B_1 \setminus \{x_1 = 0\}$  in the classical sense as well as in the viscosity sense. But  $u$  does not satisfy  $-\Delta u = 0$  in  $B_1$  in the viscosity sense.

In Theorem 2.3, the conditions (A1) and (A2) are quite natural, and it is necessary to assume (A3) since the function  $u \equiv 0$  must be a solution to (2.1). However, the condition (A4) seems to be complicated and artificial. For the particular case that  $F$  can be expressed as  $F(x, r, q, X) = \tilde{F}(q, X)$  or  $\tilde{F}(q, X) + f(r)$ , the hypotheses can be simplified as follows.

**Corollary 2.4.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$ . We suppose that  $\tilde{F} = \tilde{F}(q, X)$  and  $f = f(r)$  satisfy the following conditions.*

(B1)  $\tilde{F}$  is a continuous function defined in  $\mathbb{R}^n \times \mathbb{S}^{n \times n}$  and  $f$  is a continuous function defined in  $\mathbb{R}$ .

(B2)  $\tilde{F}$  is degenerate elliptic.

(B3)  $\tilde{F}(0, O) + f(0) = 0$ .

If  $u \in C^1(\Omega)$  is a viscosity solution to

$$\tilde{F}(Du, D^2u) + f(u) = 0 \quad (2.14)$$

in  $\Omega \setminus u^{-1}(0)$ , then  $u$  is a viscosity solution to (2.14) in the whole domain  $\Omega$ .

For parabolic equations (2.2), we can also define the notion of viscosity solutions and obtain the removability result similar to Theorem 2.3.

**Theorem 2.5.** *Let  $\mathcal{O}$  be a domain in  $\mathbb{R} \times \mathbb{R}^n$ . We suppose that the conditions given below are satisfied.*

(C1)  $F$  is a continuous function defined in  $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n \times n}$ .

(C2)  $F$  is degenerate elliptic.

(C3)  $F(t, x, 0, 0, O) = 0$  for every  $(t, x) \in \mathcal{O}$ .

(C4) *There exists a constant  $\alpha > 2$  such that for every compact subset  $K \Subset \mathcal{O}$  we can find positive constants  $\varepsilon, C$  and a continuous, non-decreasing function  $\omega_K : [0, \infty) \rightarrow [0, \infty)$  which satisfy  $\omega_K(0) = 0$  and the following:*

$$\begin{aligned} & F(t', y, s, j|x - y|^{\alpha-2}(x - y), Y) - F(t, x, r, j|x - y|^{\alpha-2}(x - y), X) \\ & \leq \omega_K(|t - t'| + |r - s| + j|x - y|^{\alpha-1} + |x - y|) \end{aligned} \quad (2.15)$$

whenever  $(t, x), (t', y) \in K$ ,  $r, s \in (-\varepsilon, \varepsilon)$ ,  $j \geq C$ ,  $X, Y \in \mathbb{S}^{n \times n}$  and

$$\begin{aligned} -3j(\alpha - 1)|x - y|^{\alpha-2}I_{2n} & \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \\ & \leq 3j(\alpha - 1)|x - y|^{\alpha-2} \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} \end{aligned} \quad (2.16)$$

holds.

If  $u \in C^1(\mathcal{O})$  is a viscosity solution to (2.2) in  $\mathcal{O} \setminus u^{-1}(0)$ , then  $u$  is a viscosity solution to (2.2) in the whole domain  $\mathcal{O}$ .

**Remark 2.2.** For  $F$  of the form  $\tilde{F}(q, X) + f(r)$ , a level set of a viscosity solution to (2.2) is always removable if we assume the continuity of  $\tilde{F}$  and  $f$ , the degenerate ellipticity of  $\tilde{F}$ , and  $\tilde{F}(0, O) + f(0) = 0$  only, as in the elliptic case.

**Example 2.1.** Utilizing Theorem 2.3 or Corollary 2.4, and Theorem 2.5, one sees that our removability results can be applied to many well-known equations. Here are the examples.

- (i) Laplace equation  $-\Delta u = 0$ , cf. [1, 8, 15].
- (ii) The heat equation  $u_t - \Delta u = 0$ .
- (iii) Poisson equation  $-\Delta u = f(u)$ , where  $f(0) = 0$  and  $f$  is continuous, for example,  $f(u) = |u|^{p-1}u$  ( $p > 0$ ).
- (iv) Linear elliptic equations

$$-\sum_{i,j=1}^n a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^n b_i(x)D_iu(x) + c(x)u(x) = 0, \quad (2.17)$$

cf. Šabat [18].

- (v) Quasilinear elliptic equations

$$-\sum_{i,j=1}^n a_{ij}(x, u, Du)D_{ij}u(x) + b(x, u, Du) = 0, \quad (2.18)$$

such as the minimal surface equation  $-\operatorname{div}(Du/\sqrt{1+|Du|^2}) = 0$ ,  $p$ -Laplace equation  $-\Delta_p u := -\operatorname{div}(|Du|^{p-2}Du) = 0$  ( $p \geq 2$ ) and  $\infty$ -Laplace equation  $\sum_{i,j=1}^n D_iu D_ju D_{ij}u = 0$ , cf. Juutinen and Lindqvist [13]. We note that our result does not contain theirs, but that is because they utilize the quasilinear nature of the equation.

- (vi) Quasilinear parabolic equations, such as  $p$ -Laplace diffusion equation  $u_t - \Delta_p u = 0$  ( $p > 2$ ).
- (vii) Pucci's equation, which is an important example of fully nonlinear uniformly elliptic equation,

$$-\mathcal{M}_{\lambda,\Lambda}^+(D^2u) = f(u), \quad -\mathcal{M}_{\lambda,\Lambda}^-(D^2u) = f(u), \quad (2.19)$$

where  $\mathcal{M}_{\lambda,\Lambda}^+$ ,  $\mathcal{M}_{\lambda,\Lambda}^-$  are the so-called Pucci extremal operators with parameters  $0 < \lambda \leq \Lambda$  defined by

$$\mathcal{M}_{\lambda,\Lambda}^+(X) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \quad \mathcal{M}_{\lambda,\Lambda}^-(X) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i, \quad (2.20)$$

for  $X \in \mathbb{S}^{n \times n}$  (see [2, 16]). Here  $e_1, \dots, e_n$  are the eigenvalues of  $X$ .

(viii) Monge-Ampère equation

$$\det D^2 u = f(u). \quad (2.21)$$

When we are concerned with (2.21), we look for solutions in the class of convex functions. It is known that the equation (2.21) is not elliptic on all  $C^2$  functions; it is degenerate elliptic for only  $C^2$  *convex* functions. In this case, the condition (A2) is not satisfied. However, modifying our argument below appropriately, one can also apply Theorem 2.3 to (2.21) and obtain the removability result.

(ix) The parabolic Monge-Ampère equation  $u_t - (\det D^2 u)^{1/n} = 0$ .

(x)  $k$ -Hessian equation

$$F_k[u] = S_k(\lambda_1, \dots, \lambda_n) = f(u), \quad (2.22)$$

where  $\lambda = (\lambda_1, \dots, \lambda_n)$  denotes the eigenvalues of  $D^2 u$  and  $S_k$  ( $k = 1, \dots, n$ ) denotes the  $k$ -th elementary symmetric function, that is,

$$S_k(\lambda) = \sum \lambda_{i_1} \cdots \lambda_{i_k}, \quad (2.23)$$

where the sum is taken over increasing  $k$ -tuples,  $1 \leq i_1 < \cdots < i_k \leq n$ . Thus  $F_1[u] = \Delta u$  and  $F_n[u] = \det D^2 u$ , which we have seen before. This equation has been intensively studied, see for example [3, 23, 24, 25].

(xi) Gauss curvature equation

$$\det D^2 u = f(u) (1 + |Du|^{(n+2)/2}). \quad (2.24)$$

(xii) Gauss curvature flow equation  $u_t - \det D^2 u / (1 + |Du|^2)^{(n+1)/2} = 0$ .

(xiii)  $k$ -curvature equation

$$H_k[u] = S_k(\kappa_1, \dots, \kappa_n) = f(u), \quad (2.25)$$



where  $\kappa_1, \dots, \kappa_n$  denote the principal curvatures of the graph of the function  $u$ , that is, namely, the eigenvalues of the matrix

$$D \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{1}{\sqrt{1 + |Du|^2}} \left( I - \frac{Du \otimes Du}{1 + |Du|^2} \right) D^2u, \quad (2.26)$$

and  $S_k$  is the  $k$ -th elementary symmetric function. The mean, scalar and Gauss curvature equation correspond respectively to the special cases  $k = 1, 2, n$  in (2.25). For the classical Dirichlet problem for  $k$ -curvature equations in the case that  $2 \leq k \leq n - 1$ , see for instance [4, 10, 22].

We could also prove the removability of a level set for solutions to the singular equations such as  $p$ -Laplace diffusion equation where  $1 < p < 2$ . See [21] for details.

In the final part of this section, we give a sketch of the proof of Theorem 2.3. This is divided into two parts.

**Step 1.** Removability of the set  $\{x \in \Omega \mid u(x) = 0, Du(x) \neq 0\}$

Let  $x_0$  be a point in  $\{x \in \Omega \mid u(x) = 0, Du(x) \neq 0\}$ . Then it follows from the implicit function theorem that the level set  $\{u = 0\}$  is locally a  $C^1$  hypersurface.

Let  $\varphi \in C^2(\Omega)$  be any function such that  $x_0$  is a maximum point of  $u - \varphi$ . We want to show (2.8). For this purpose, we add an appropriate small perturbation  $\psi_\delta$  to  $\varphi$  such that  $\psi_\delta \rightarrow 0$  in  $C^2(\Omega)$  as  $\delta \rightarrow +0$  and that the maximum of  $u - (\varphi + \psi_\delta)$  attains at a point  $x_\delta$  which lies in  $\{u \neq 0\}$ . It follows from the definition of the viscosity subsolution that

$$F(x_\delta, u(x_\delta), D(\varphi + \psi_\delta)(x_\delta), D^2(\varphi + \psi_\delta)(x_\delta)) \leq 0. \quad (2.27)$$

Letting  $\delta \rightarrow +0$ , we can show that  $x_\delta \rightarrow x_0$  and obtain (2.8).

**Step 2.** Removability of the set  $\{x \in \Omega \mid u(x) = 0, Du(x) = 0\}$

In this case, we can prove that in the definition of viscosity solutions, we require no testing at all at the points where the gradient of  $u$  vanishes under our assumptions (i.e., if a test function  $\varphi$  and a “touching point”  $x_0$  satisfy  $D\varphi(x_0) = 0$ , then  $F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0$  ( $\geq 0$ ) must hold.).

### 3 Main results

In this section, we state our Král type removability result for  $k$ -Hessian equations

$$F_k[u] = S_k(\lambda_1, \dots, \lambda_n) = 0, \quad (3.1)$$

where  $\lambda_1, \dots, \lambda_n$  denotes the eigenvalues of  $D^2u$ , and  $k$ -curvature equations

$$H_k[u] = S_k(\kappa_1, \dots, \kappa_n) = 0, \quad (3.2)$$

where  $\kappa_1, \dots, \kappa_n$  denote the principal curvatures of the graph of the function  $u$ .

To deal with our problem, we consider the class of *generalized solutions* instead of that of viscosity solutions. The notion of generalized solutions gives a new framework for the study of  $k$ -Hessian equations  $F_k[u] = \psi$  and  $k$ -curvature equations  $H_k[u] = \psi$  where  $\psi$  is a Borel measure. It is introduced by Colesanti and Salani [5] and Trudinger and Wang [23, 24, 25] for  $k$ -Hessian equations and by the author [20] for  $k$ -curvature equations. Here we only focus on the case of  $k$ -Hessian equations (3.1). We can treat the case of  $k$ -curvature equations (3.2), see [19] for details.

Let  $\Omega \subset \mathbb{R}^n$  be a domain. We define the set  $\Phi^k(\Omega)$  as follows:

$$\Phi^k(\Omega) = \{u : \Omega \rightarrow [-\infty, \infty) \mid u \text{ is a viscosity subsolution to } F_k[u] = 0.\}. \quad (3.3)$$

We omit the proof of the following proposition.

**Proposition 3.1.** (i)  $\Phi^1(\Omega) \supset \Phi^2(\Omega) \supset \dots \supset \Phi^n(\Omega)$ .

(ii)  $\Phi^1(\Omega)$  is a set of subharmonic functions on  $\Omega$ , and  $\Phi^n(\Omega)$  is a set of convex functions on  $\Omega$ .

The important fact is that for  $u \in \Phi^k(\Omega)$ , we can define  $F_k[u]$  as a Borel measure, which is well-known for the cases  $k = 1$  and  $k = n$ .

**Theorem 3.2.** [23] Let  $\Omega$  be an open convex bounded set in  $\mathbb{R}^n$ , and let  $u \in \Phi^k(\Omega)$ . Then there exist a unique nonnegative Borel measure  $\sigma_k(u; \cdot)$  such that the following properties hold:

(i) If  $u \in C^2(\Omega)$ , then for every Borel subset  $\eta$  of  $\Omega$ ,

$$\sigma_k(u; \eta) = \int_{\eta} F_k[u](x) dx. \quad (3.4)$$

(ii) If  $u, u_i \in \Phi^k(\Omega)$  ( $i \in \mathbb{N}$ ) satisfy  $u_i \rightarrow u$  in  $L^1_{loc}(\Omega)$ , then

$$\sigma_k(u_i; \cdot) \rightharpoonup \sigma_k(u; \cdot) \quad (\text{weakly}). \quad (3.5)$$

**Example 3.1.** Let  $B_1$  be a unit ball in  $\mathbb{R}^n$  and  $\alpha$  be a positive constant.

(1) Let  $u_1(x) = \alpha|x|$ . Then

$$F_n[u_1] = \omega_n \alpha^n \delta_0. \quad \text{in } B_1, \quad (3.6)$$

where  $\omega_n$  denotes the volume of the unit ball in  $\mathbb{R}^n$ , and  $\delta_0$  is the Dirac measure at 0.

(2) Let  $u_2(x) = \alpha\sqrt{x_1^2 + \cdots + x_k^2}$ , where  $x = (x_1, \dots, x_n)$ . Then

$$F_k[u_2] = \omega_k \alpha^k \mathcal{L}^{n-k}[T] \quad \text{in } B_1, \quad (3.7)$$

where  $\omega_k$  denotes the  $k$ -dimensional measure of the unit ball in  $\mathbb{R}^k$  and  $T = \{(x_1, \dots, x_n) \in B_1 \mid x_1 = \cdots = x_k = 0\}$ .

The definition of generalized solutions of curvature equations is given as follows:

**Definition 3.3.** Let  $\Omega$  be a domain in  $\mathbb{R}^n$ , let  $\nu$  be a nonnegative finite Borel measure on  $\Omega$ .  $u \in \Phi^k(\Omega)$  is said to be a *generalized solution* of

$$F_k[u] = \nu \quad \text{in } \Omega, \quad (3.8)$$

if it holds that

$$\sigma_k(u; \eta) = \nu(\eta) \quad (3.9)$$

for every Borel subset  $\eta$  of  $\Omega$ .

The following proposition indicates that the notion of generalized solutions is weaker (hence wider) than that of viscosity solutions in some sense.

**Proposition 3.4.** Suppose  $\psi \in C^0(\Omega)$  is a nonnegative function and set  $\nu = \psi dx$ . If  $u$  is a viscosity solution to  $F_k[u] = \psi$  in  $\Omega$ , then it is a generalized solution to  $F_k[u] = \nu$  in  $\Omega$ .

Colesanti and Salani [5] give the characterization of  $\sigma_k(u; \cdot)$  for a convex function  $u$  defined in a convex domain  $\Omega$  (we note that  $u \in \Phi^k(\Omega)$  due to Proposition 3.1(i)). For  $x \in \Omega$ ,  $\partial u(x)$  denotes the subdifferential of  $u$  at  $x$  (if  $u$  is  $C^1$  at  $x$ , then  $\partial u(x) = \{Du(x)\}$ ). For  $\rho > 0$  and a Borel subset  $\eta$  of  $\Omega$ , we set

$$P_\rho(u; \eta) := \{z \in \mathbb{R}^n \mid z = x + \rho v, \ x \in \eta, \ v \in \partial u(x)\}. \quad (3.10)$$

Then the following equality holds:

$$\mathcal{L}^n(P_\rho(u; \eta)) = \sum_{j=0}^n \sigma_j(u; \eta) \rho^j. \quad (3.11)$$

Here we define  $\sigma_0(u; \eta) := \mathcal{L}^n(\eta)$ .

Now we state the Král type removability result for  $k$ -Hessian equations (3.1).

**Theorem 3.5.** Let  $\Omega \subset \mathbb{R}^n$  be a domain,  $u \in C^1(\Omega)$  and  $E$  a subset of  $\mathbb{R}$ . We suppose that each compact subset  $F$  of  $E$  is at most countable and that for every compact set  $K \Subset \Omega$ ,

$$\sup\{|Du(x) - Du(y)| \mid x, y \in K, |x - y| \leq \delta\} = o(\delta^{(k-1)/k}) \quad (\text{as } \delta \rightarrow +0). \quad (3.12)$$

If  $u$  is a generalized solution to (3.1) in  $\Omega \setminus u^{-1}(E)$ , then it is a generalized solution to (3.1) in the whole domain  $\Omega$ .

We can obtain the removability result similar to Theorem 3.5 for the  $k$ -curvature equation (3.2).

## 4 Sketch of the proof of Theorem 3.5

In this section, we give a sketch of the proof of our main theorem, Theorem 3.5. We can prove the removability of  $u^{-1}(E) \cap \{x \in \Omega \mid Du(x) = 0\}$  in a similar way to Step 2 of Theorem 2.3.

We fix a point  $x_0$  in  $u^{-1}(E) \cap \{x \in \Omega \mid Du(x) \neq 0\}$ . It follows from the implicit function theorem that for some small neighborhood  $U_1, U_2$  of  $x_0$  ( $U_1 \Subset U_2$ ), the Hausdorff dimension of  $A := U_1 \cap u^{-1}(E)$  is  $n - 1$ . We set

$$\psi(\delta) = \sup\{|Du(x) - Du(y)| \mid x, y \in \overline{U_2}, |x - y| \leq \delta\}. \quad (4.1)$$

By the assumption, we get that  $\psi(\delta) = o(\delta^{(k-1)/k})$ , i.e.,  $\delta^{n-k}\psi(\delta)^k = o(\delta^{n-1})$ .

We fix  $\varepsilon > 0$ . Then from the fact stated above, there exists countable balls  $\{B_{r_i}(x_i)\}_{i=1}^\infty$  such that

$$A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i) \subset U_2 \quad \text{and} \quad \sum_{i=1}^\infty r_i^{n-k}\psi(r_i)^k < \varepsilon. \quad (4.2)$$

We can show that

$$P_\rho(u; B_{r_i}(x_i)) \subset B_{r_i+\rho\psi(r_i)}(x_i + \rho Du(x_i)). \quad (4.3)$$

Indeed, taking any  $z \in P_\rho(u; B_{r_i}(x_i))$  we obtain

$$|z - (x_i + \rho Du(x_i))| \leq |y - x_i| + \rho|Du(y) - Du(x_i)| < r_i + \rho\psi(r_i). \quad (4.4)$$

for some  $y \in B_{r_i}(x_i)$ . Therefore, it follows from (3.11) that

$$\begin{aligned} \sigma_k(u; B_{r_i}(x_i))\rho^k &\leq \sum_{j=0}^n \sigma_j(u; B_{r_i}(x_i))\rho^j \\ &= \mathcal{L}^n(P_\rho(u; B_{r_i}(x_i))) \\ &\leq \mathcal{L}^n(B_{r_i+\rho\psi(r_i)}(x_i + \rho Du(x_i))) \\ &= \omega_n(r_i + \rho\psi(r_i))^n. \end{aligned} \quad (4.5)$$

Now we put  $\rho := r_i/\psi(r_i)$ . We obtain that

$$\sigma_k(u; B_{r_i}(x_i)) \leq 2^n \omega_n r_i^{n-k} \psi(r_i)^k. \quad (4.6)$$

It holds that

$$\sigma_k(u; A) \leq \sum_{i=1}^{\infty} \sigma_k(u; B_{r_i}(x_i)) \leq \sum_{i=1}^{\infty} 2^n \omega_n r_i^{n-k} \psi(r_i)^k = 2^n \omega_n \varepsilon. \quad (4.7)$$

Thus we have  $\sigma_k(u; A) = 0$  due to the arbitrariness of  $\varepsilon$ . The proof that  $u$  satisfies  $F_k[u] = 0$  in the whole domain  $\Omega$  is complete.

**Remark 4.1.** For the case of  $k = 1$  (Laplace equation), the convexity assumption of  $u$  can be removed so that we get the same removability result as Král's.

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## References

- [1] E.F. Beckenbach, *On characteristic properties of harmonic functions*, Proc. Amer. Math. Soc. **3** (1952), 765–769.
- [2] L. Caffarelli and X. Cabre, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, 43, American Mathematical Society, Providence, 1995.
- [3] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III. Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261–301.
- [4] ———, *Nonlinear second-order elliptic equations, V. The Dirichlet problem for Weingarten hypersurfaces*, Comm. Pure Appl. Math. **42** (1988), 47–70.
- [5] A. Colesanti and P. Salani, *Generalised solutions of Hessian equations*, Bull. Austral. Math. Soc. **56** (1997), 459–466.
- [6] M.G. Crandall, *Viscosity solutions: a primer*, Viscosity solutions and applications (Montecatini Terme, 1995), Lecture Notes in Math., 1660, Springer, Berlin, 1997, pp. 1–43.

- [7] M.G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67.
- [8] J.W. Green, *Functions that are harmonic or zero*, Amer. J. Math. **82** (1960), 867–872.
- [9] H. Ishii and P.-L. Lions, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations*, J. Differential Equations **83** (1990), 26–78.
- [10] N.M. Ivochkina, *The Dirichlet problem for the equations of curvature of order  $m$* , Leningrad Math. J. **2** (1991), 631–654.
- [11] R. Jensen, *Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations*, Indiana Univ. Math. J. **38** (1989), 629–667.
- [12] P. Juutinen and P. Lindqvist, *A theorem of Radó's type the solutions of a quasi-linear equation*, Math. Res. Lett. **11** (2004), 31–34.
- [13] ———, *Removability of a level set for solutions of quasilinear equations*, Comm. Partial Differential Equations **30** (2005), 305–321.
- [14] T. Kilpeläinen, *A radó type theorem for  $p$ -harmonic functions in the plane*, Electron. J. Differential Equations **9** (1994), 1–4.
- [15] J. Král, *Some extension results concerning harmonic functions*, J. London Math. Soc. (2) **28** (1983), 62–70.
- [16] C. Pucci, *Operatori ellittici estremanti*, Ann. Mat. Pura Appl. (4) **72** (1966), 141–170.
- [17] T. Radó, *Über eine nicht fortsetzbare Riemannsche Mannigfaltigkeit*, Math. Z. **20** (1924), 1–6.
- [18] A.B. Šabat, *On a property of solutions of elliptic equations of second order*, Soviet Math. Dokl. **6** (1965), 926–928.
- [19] K. Takimoto, *Some removability results for solutions to fully nonlinear equations*, in preparation.
- [20] ———, *Generalized solutions of curvature equations*, Nonlinear Anal. **67** (2007), 1275–1287.

- [21] ———, *Radó type removability result for fully nonlinear equations*, Differential Integral Equations **20** (2007), 939–960.
- [22] N.S. Trudinger, *The Dirichlet problem for the prescribed curvature equations*, Arch. Ration. Mech. Anal. **111** (1990), 153–179.
- [23] N.S. Trudinger and X.J. Wang, *Hessian measures I*, Topol. Methods Nonlinear Anal. **10** (1997), 225–239.
- [24] ———, *Hessian measures II*, Ann. of Math. **150** (1999), 579–604.
- [25] ———, *Hessian measures III*, J. Funct. Anal. **193** (2002), 1–23.